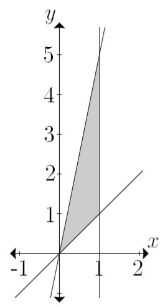
Snow College Mathematics Contest

key

April 4, 2017 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Find the volume of the 3-D shape formed by revolving the region bounded by $y = 5x$, $y = x$, and $x = 1$ around the y-axis.



(A) $\frac{8\pi}{3}$
(B) 24π
(C) $\frac{4\pi}{3}$
(D) 16π
(E) $\frac{16\pi}{3}$

SOLN Use the shell method.
$V = 2\pi \int_0^1 (5x - x)x \, dx = 2\pi \left[\frac{4}{3}x^3\right]_0^1$ □

2. If $a + b = 3$ and $a^2 + b^2 = 89$, then what is $a^3 + b^3$?

(A) 307
(B) 347
(C) 387
(D) 507
(E) Not possible to determine

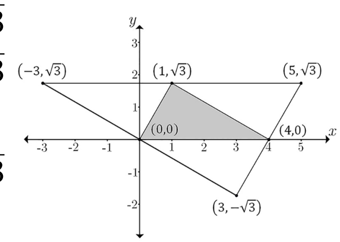
SOLN $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ so $a^3 + b^3 = 3(89 - ab)$. Note $9 = (a+b)^2 = (a^2 + 2ab + b^2)$, so $ab = -40$. □

3. What is the value of $\log_2 \left(7^{-\log_7 \frac{1}{8}}\right)$?

(A) -3
(B) $\frac{1}{8}$
(C) $\frac{1}{3}$
(D) 3
(E) 8

SOLN $\log_2 \left(7^{\log_7 \left(\frac{1}{8}\right)^{-1}}\right) = \log_2 (\frac{1}{8})^{-1}$
$= \log_2 8 = \log_2 2^3$ □

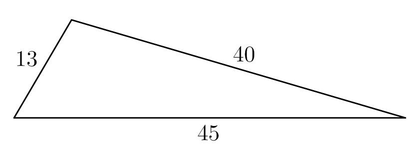
4. How many parallelograms can be formed with these three points as vertices, and what is the area of each? $(0, 0)$, $(1, \sqrt{3})$, $(4, 0)$



(A) $1,\ A = 5\sqrt{3}$
(B) $3,\ A = 4\sqrt{3}$
(C) $4,\ A = 10$
(D) $2,\ A = 5\sqrt{3}$
(E) $2,\ A = 8$

SOLN The three given vertices form a triangle; they also are midpoints of the sides of a larger triangle. The line segment connecting any pair of these three points is parallel to the other side of the larger triangle. The shaded triangle plus any one of the other three triangles makes a parallelogram. All four triangles have the same area. This would be true for any three non-collinear points. □

5. Find the area of the triangle.



(A) 252
(B) 260
(C) 298
(D) 312
(E) 333

SOLN Heron's formula: for sides a, b, c, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is semi-perimeter. $P = 98 \Rightarrow s = 49$
$A = \sqrt{49(49-13)(49-40)(49-45)}$
$= \sqrt{49(36)(9)(4)} = 7 \cdot 6 \cdot 3 \cdot 2$ □

6. The operation $\oplus$, called *reciprocal sum*, is useful in many areas of physics. $x = a \oplus b$ means $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$. Compute the following.
$4 \oplus 2 \oplus 4 \oplus 3 \oplus 4 \oplus 4 \oplus 2 \oplus 3 \oplus 2 \oplus 4 \oplus 4 \oplus 3$

(A) $\frac{3}{4}$
(B) $\frac{1}{2}$
(C) $\frac{3}{8}$
(D) $\frac{1}{8}$
(E) $\frac{1}{4}$

SOLN $\oplus$ is associative and commutative, so the computation can be rewritten $(2\oplus2\oplus2)\oplus(3\oplus3\oplus3)\oplus(4\oplus4\oplus4\oplus4\oplus4\oplus4)$. Furthermore, $1/(x_1 \oplus x_2 \oplus \ldots \oplus x_n) = \frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}$. So $2\oplus2\oplus2 = \frac{1}{3/2} = \frac{2}{3}$, $3\oplus3\oplus3 = 1$, and $4\oplus4\oplus4\oplus4\oplus4\oplus4 = \frac{1}{6/4} = \frac{2}{3}$. Answer: $1/(\frac{3}{2}+1+\frac{3}{2}) = \frac{1}{4}$ □

7. Both the surface area and volume of a sphere increase as the radius increases. A increases faster than V when r is small, but V increases faster than A when r is large. At what value of r does the rate of increase in V overtake the rate of increase in A?

(A) $\frac{2}{3}$
(B) 1
(C) $\frac{4}{3}$
(D) 2
(E) $\frac{8}{3}$

SOLN $V = \frac{4}{3}\pi r^3 \implies dV/dr = 4\pi r^2$
$A = 4\pi r^2 \implies dA/dr = 8\pi r$
Equating the two gives $r = 2$. □

8. In base three, what is three times 2102?

(A) 20210
(B) 2110
(C) 12100
(D) 11201
(E) 21020

SOLN Multiplying any number by the base simply moves all digits left one place. $1011_{\text{two}} \times 10_{\text{two}} = 10110_{\text{two}}$ □

9. An earthquake generates both *primary* and *secondary* seismic waves. In a certain material the longitudinal P-waves have a speed of 5000 m/s and the transverse S-waves have a speed of 3000 m/s. If a seismometer detects P-waves in that material 4 s before the S-waves, how far away is the epicenter?

(A) 20 km
(B) 30 km
(C) 40 km
(D) 50 km
(E) 60 km

SOLN Both travel the same distance.
$d = v_S t_S = v_P t_P \implies t_P = t_S \frac{v_S}{v_P}$
$\Delta t = t_S - t_P = t_S - t_S\frac{v_S}{v_P} = t_S(1 - \frac{v_S}{v_P})$
$t_S = (4\,\text{s})/(1 - 0.6) = 10\,\text{s}$
$d = v_S t_S = (3000\,\frac{\text{m}}{\text{s}})(10\,\text{s}) = 30\,\text{km}$ □

10. For a matrix A, an eigenvalue-eigenvector pair is a constant λ and corresponding nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Which is an eigenvalue λ of this matrix?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

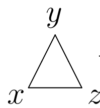
(A) -1
(B) 0
(C) 1
(D) 2
(E) 4

SOLN Characteristic eqn. of the matrix:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda = 0$$

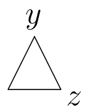
So the eigenvalues are $\lambda = 0, 5$ with corresponding eigenvectors
$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Also, $\det(A) = \prod \lambda_i$ and $\text{Tr}(A) = \sum \lambda_i$.
$\lambda_1 \cdot \lambda_2 = 0$ and $\lambda_1 + \lambda_2 = 5 \implies$
$\lambda_1, \lambda_2 = 0, 5$ □

11. Definition of the *triangle of power* notation:



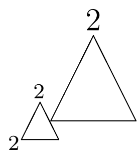
$\Leftrightarrow x^y = z \Leftrightarrow \sqrt[y]{z} = x \Leftrightarrow \log_x z = y$

Any of x, y, z is equivalent to the triangle of power with that number missing, e.g.,



$= x$

Find the value of the following.



(A) 2

(B) $\sqrt{8}$

(C) 3

(D) 4

(E) 16

SOLN $2^2 = 4,\ 4^2 = 16$

https://www.youtube.com/watch?v=sULa9Lc4pck □

12. The city council has changed the numbering scheme for the 200 houses on Elm Street. They will be renumbered with the natural numbers from 1 through 200. A city worker is given a box of 1000 metal numbers, 100 of each digit, and told to distribute new house numbers in order starting with 1 Elm Street. What is the first address for which he will not have the correct digits?

(A) 137 Elm Street

(B) 163 Elm Street

(C) 172 Elm Street

(D) 191 Elm Street

(E) 199 Elm Street

SOLN The first metal 1 will be used for address 1, the next eleven 1s for addresses 10–19, the next eight for 21–91, the next eleven for 100–109, the next twenty-one for 110–119, and another eleven each for 120–129, 130–139, 140–149, 150–159, and the last four for 160–162. □

13. What is the smallest positive integer n such that $1 + 2 + 3 + \ldots + n > 5000$?

(A) 90

(B) 99

(C) 100

(D) 101

(E) 110

SOLN The sum of the first n positive integers is $\frac{n(n+1)}{2}$. The solution is the first positive n for which $n(n+1) > 10000$. When is the product of consecutive positive integers first larger than $10000 = 100^2$? Clearly, $99 \cdot 100 < 100 \cdot 100 < 100 \cdot 101$; therefore $n = 100$. □

14. A *metric space* is a set A where we have a notion of distance, i.e., if $a, b \in A$, then $d(a, b)$ is the "distance" between a and b. The distance function must satisfy these conditions:

(i) $d(a, b) \geq 0$ for all $a, b \in A$ positivity

(ii) $d(a, b) = 0$ iff $a = b$ non-degenerate

(iii) $d(a, b) = d(a, b)$ symmetry

(iv) $d(a, c) \leq d(a, b) + d(b, c)$ triangle inequality

Does $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ define a metric on $\mathbb{R}^2$ (the xy-plane)?

(A) No; condition (i) is not met.

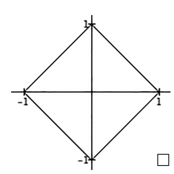
(B) No; condition (ii) is not met.

(C) No; condition (iii) is not met.

(D) No; condition (iv) is not met.

(E) Yes; all conditions are met.

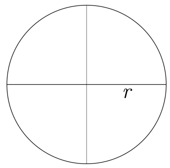
SOLN This is called the *taxicab metric*. The "unit circle" in a metric space in $\mathbb{R}^2$ is the locus $\{(x, y) \mid d((0, 0), (x, y)) = 1\}$. Here is the unit circle for the taxicab metric:

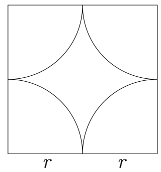


□

15. What is the ratio of the area of a circle to the area of the figure created by flipping each quarter circle around its chord?

(A) $\frac{3\pi}{4}$
(B) $\pi + 1$
(C) π
(D) $\pi - 1$
(E) $\frac{\pi}{4-\pi}$

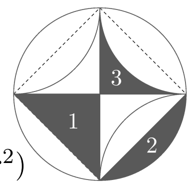




SOLN $A_\diamondsuit = A_\square - A_\bigcirc = (2r)^2 - \pi r^2$

$$\frac{A_\bigcirc}{A_\diamondsuit} = \frac{\pi r^2}{4r^2 - \pi r^2} = \frac{\pi}{4-\pi}$$

Method 2:



$A_1 = \frac{1}{2}r^2$
$A_2 = \frac{1}{4}\pi r^2 - \frac{1}{2}r^2$
$A_3 = \frac{1}{2}r^2 - (\frac{1}{4}\pi r^2 - \frac{1}{2}r^2)$
$= r^2 - \frac{1}{4}\pi r^2 = r^2(1 - \frac{\pi}{4})$

$$\frac{A_\bigcirc}{4A_3} = \frac{\pi r^2}{4r^2(1-\frac{\pi}{4})} = \frac{\pi}{4-\pi}$$

Method 3: use an integral to find A_3:

$$\int_0^r \left(-\sqrt{r^2-(x-r)^2} + r\right) \mathrm{d}x$$

Substitute $u = x - r, \quad \mathrm{d}u = \mathrm{d}x$.

$$\int_0^r r \;\mathrm{d}x - \int_{-r}^0 \sqrt{r^2-u^2}\;\mathrm{d}u$$

$$= [rx]_0^r - \left[\frac{u\sqrt{r^2-u^2}}{2} + \frac{r^2}{2}\sin^{-1}\frac{u}{r}\right]_{-r}^0$$

$$= r^2 - \frac{r^2}{2}\frac{\pi}{2}$$

□

16. How many times does 12 appear in the output of this BASIC computer program?

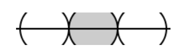
(A) 2
(B) 3
(C) 4
(D) 5
(E) 6

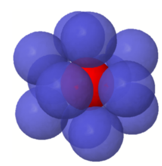
```
10 for i = 1 to 4
20   for j = 1 to i
30    for k = 1 to j
40     print i*j*k
50    next k
60   next j
70 next i
```

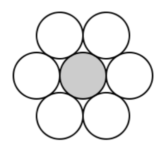
SOLN $i \cdot j \cdot k$, where $i \geq j \geq k$, i.e., when $\mathtt{i} = 3,\ \mathtt{j} = 2,\ \mathtt{k} = 2$ and when $\mathtt{i} = 4,\ \mathtt{j} = 3,\ \mathtt{k} = 1$. □

17. In geometry, a *kissing number* is the number of non-overlapping unit spheres that can be arranged such that they each touch a given unit sphere. In 1-D, the kissing number is 2. In 3-D the kissing number is 12. What is the kissing number in 2-D?

(A) 4
(B) 5
(C) 6
(D) 7
(E) 8







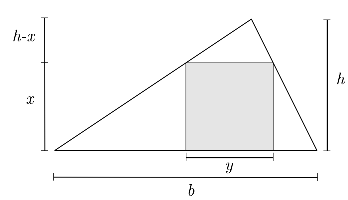
SOLN

https://en.wikipedia.org/wiki/Kissing_number_problem

There is almost enough room left over in 3-D to fit in a 13th, but not quite. □

18. Given a triangle of base b and altitude h, a rectangle of height x is inscribed in the triangle with its base in the base of the triangle. What is the area of the rectangle?

(A) $\frac{bx}{h}(h-x)$
(B) $\frac{hx}{b}(b-x)$
(C) $\frac{bx}{h}(h-2x)$
(D) $x(b-x)$
(E) $x(h-x)$



SOLN Designate the base of the rectangle by y. Because of similar triangles, $\frac{h-x}{y} = \frac{h}{b}$. $\therefore A = xy = x\frac{b}{h}(h-x)$. □

19. Simplify $\sqrt{\sqrt[0.06]{2^{0.12}}}$

(A) 1

(B) $\sqrt{2}$

[(C)] 2

(D) $2^{0.0384}$

(E) 4

SOLN Writing with rational exponents gives $((2^{0.12})^{1/0.06})^{1/2} = 2^{(0.12)(1/0.06)(1/2)} = 2^1 = 2$. □

20. Find the remainder for the quotient.

$$\frac{2x^3 - 9x^2 + 6x - 1}{2x - 1}$$

(A) $2x - 1$

(B) $2x^2 - 8x + 2$

[(C)] 0

(D) $\frac{1}{2}$

(E) $2x^3 - 9x^2 + 3$

SOLN The remainder can be found using polynomial long division or synthetic division using $x = 1/2$. Synthetic division produces

	2	−9	6	−1
$\frac{1}{2}$		1	−4	1
	2	−8	2	0

showing that the remainder is 0. □

21. A cube with side length $\sqrt{6}$ cm is cut revealing pyramid $ABCD$ as shown. Find the surface area of the pyramid.

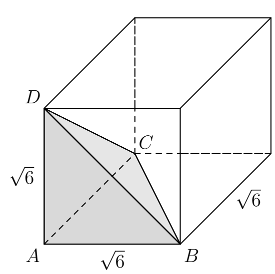
(A) 12

(B) 18

(C) $4\sqrt{6}$

[(D)] $9 + 3\sqrt{3}$

(E) $54 + 3\sqrt{3}$



SOLN Three faces are right triangles each with area $\frac{1}{2}\sqrt{6}\sqrt{6} = 3$. ΔBCD is equilateral with side lengths $\sqrt{12} = 2\sqrt{3}$ for a total area of $3\sqrt{3}$. Total area is $3(3) + 3\sqrt{3} = 9 + 3\sqrt{3}$. □

22. In a drawer are 6 black, 2 gray, and 2 tan socks. What is the probability of blindly pulling out (without replacement) four black socks in a row?

[(A)] $\frac{1}{14}$

(B) $\frac{1}{7}$

(C) $\frac{1}{4}$

(D) $\frac{2}{5}$

(E) $\frac{3}{5}$

SOLN To find the probability, multiply $\left(\frac{6}{10}\right)\left(\frac{5}{9}\right)\left(\frac{4}{8}\right)\left(\frac{3}{7}\right) = \left(\frac{3}{5}\right)\left(\frac{5}{9}\right)\left(\frac{1}{2}\right)\left(\frac{3}{7}\right)$ □

23. Here is an **incorrect** "proof" that $1 = -1$. Which step of the proof is incorrect?

$1 = \sqrt{1}$

(A) $1 = \sqrt{(-1)(-1)}$

[(B)] $1 = \sqrt{-1}\sqrt{-1}$

(C) $1 = i \cdot i$

(D) $1 = i^2$

(E) $1 = -1$

SOLN The property $\sqrt{AB} = \sqrt{A}\sqrt{B}$ requires that A and B are not simultaneously less than zero. □

24. Given sets $A = \{\text{evens}\}$, $B = \{\text{non-primes}\}$, and $C = \{\text{primes} < 19\}$, and universal set $U = \{0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23\}$, find the complement of $A \cup B \cup C$.

(A) $\{0, 23\}$

(B) $\{0, 10, 20\}$

[(C)] $\{23\}$

(D) $\{1, 3, 11, 13, 21, 23\}$

(E) $\emptyset$ or $\{\ \}$

SOLN $A \cup B = \{0, 2, 10, 12, 20, 21, 22\}$. Then the union of this set with C is $\{0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22\}$ the complement in U of which is $\{23\}$. □

25. In a music class, 15 students play violin, 14 play piano, and 16 play horns. Of these, 6 play piano and violin, 9 play piano and horns, and 5 play horns and violin. Four students play all three and 2 students play none. How many students are in the class?

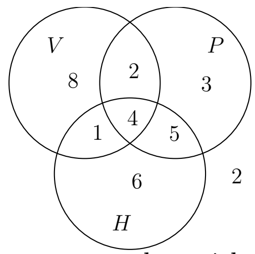
(A) 21

(B) 23

[(C)] 31

(D) 45

(E) 47



SOLN On a Venn diagram we see that with 4 students who play all three instruments there are 2 who play only violin and piano, 5 who play only piano and horns, and 1 who plays only violin and horns. From there, we see that 8 play violin only, 3 play piano only, and 6 play horns only. With the 2 who do not play any instruments, we have a total of 31. □

26. A cyclist made a 31.5 mile trip. Later, she calculated that if she had increased her average speed by 2 mph, she would have saved an hour. What was her original speed?

(A) 6 mph

[(B)] 7 mph

(C) 8 mph

(D) 9 mph

(E) 10 mph

SOLN $d = rt \implies 31.5 = rt$ or $t = 31.5/r$. With her increased speed and decreased time, the equation would be $31.5 = (r+2)(t-1) = (r+2)(31.5/r - 1)$. Multiplying by r and distributing gives $31.5r = 31.5r - r^2 + 63 - 2r$. Combining terms: $r^2 + 2r - 63 = (r-7)(r+9) = 0$; the only viable solution is $r = 7$ mph. □

27. What is the sum of the real solutions?

$$\left(\frac{x-5}{3}\right)^{x^2+x} = 1$$

[(A)] 9

(B) -1

(C) 8

(D) 1

(E) 10

SOLN For the right side to be 1, the exponent must be 0 or the base must be ± 1.
Exp $= 0$: $x(x+1) = 0 \Rightarrow x = 0, -1$
Base $= 1$: $(x-5)/3 = 1 \Rightarrow x = 8$
Base $= -1$: $(x-5)/3 = -1 \Rightarrow x = 2$ □

28. The average of a and $2b$ is 7; the average of a and $2c$ is 8. What is the average of a, b, and c?

(A) 3

(B) 4

[(C)] 5

(D) 6

(E) 9

SOLN $(a+2b)/2 = 7 \implies b = (14-a)/2$
$(a+2c)/2 = 8 \implies c = (16-a)/2$

$$\frac{a+b+c}{3} = \frac{a + \frac{14-a}{2} + \frac{16-a}{2}}{3} = \frac{30}{6}$$

Method 2: $\frac{a+2b}{2} = 7, \quad \frac{a+2c}{2} = 8 \implies \frac{2a+2b+2c}{2} = 7 + 8 \implies a + b + c = 15$ □

29. For how many integers b is the polynomial $x^2 + bx + 16$ factorable over the integers?

(A) 2

(B) 3

(C) 4

(D) 5

[(E)] 6

SOLN b can be positive or negative, but the factors of 16 must have the same sign.
$b = \{\pm 8, \pm 10, \pm 17\}$ □

30. A student committee must consist of two seniors and three juniors. Five seniors are available to serve. What is the least number of junior volunteers needed if the selectors want at least 600 different possible ways to choose the committee?

(A) 6

(B) 7

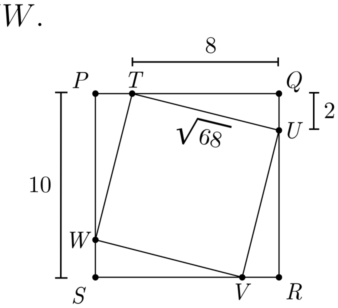
(C) 8

(D) 9

(E) 10

SOLN We have $\binom{5}{2} = {}_5C_2 = 10$ ways to choose the seniors, so we must have at least 60 ways to choose the juniors. We seek the smallest n such that $\binom{n}{3} \geq 60$. $\binom{8}{3} = 56$, and $\binom{9}{3} = 84$. □

31. Square $PQRS$ has sides of length 10. Points T, U, V, and W are chosen on sides PQ, QR, RS, and SP respectively so that $PT = QU = RV = SW = 2$. Find the area of quadrilateral $TUVW$.



(A) 48

(B) 52

(C) 56

(D) 64

(E) 68

SOLN You have a square within a square whose sides form the equal hypotenuses of 4 right triangles with legs 2 and 8. Each side of the inner square is $\sqrt{68}$. □

32. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{10}$$

Find the last two digits of element b.

(A) 15

(B) 23

(C) 31

(D) 63

(E) 07

SOLN Can be done by brute force.

$$A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix}, \ldots$$

$$A^n = \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$$

Since $2^{10} = 1024$ and $2^{10} - 1 = 1023$ □

33. Find the sum of the series:

$$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots$$

(A) $\frac{3}{2}$

(B) ∞

(C) π

(D) $\sqrt{2}$

(E) $\frac{9}{4}$

SOLN $S = \dfrac{a}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{2}{3}} = \dfrac{\frac{1}{2}}{\frac{1}{3}}$ □

34. Two straight lines have the same y-intercept and reciprocal slopes. If the first line has slope m and x-intercept a, what is the x-intercept of the other line?

(A) $\frac{m^2}{a}$

(B) $\frac{m}{a}$

(C) $\frac{1}{a}$

(D) am

$\boxed{\text{(E)}}$ am^2

SOLN The equation of the first line is $y = m(x-a)$ so the y-intercept is $-ma$. Therefore the equation of the second line is $y = \frac{1}{m}x - ma$. Setting $y=0$, $x = am^2$. From AMATYC Faculty Math League. □

35. In one game you pick three distinct numbers from 01 to 20. You win if the three numbers you pick match the three numbered balls drawn at random (order doesn't matter). What is the probability of winning?

(A) $\frac{3}{20}$

$\boxed{\text{(B)}}$ $\frac{3!\,17!}{20!}$

(C) $\frac{1}{17!}$

(D) $\frac{3!}{17!}$

(E) $\frac{20!}{3!\,17!}$

SOLN Since order doesn't matter, there are $\binom{20}{3} = \frac{20!}{3!\,17!}$ ways to draw 3 balls. For your pick of any three numbers, your chances of winning are $\frac{3!\,17!}{20!}$. □

36. Pi High School sent a team of 5 students to the Snow College Math Contest in 2015. In 2016 PHS sent the same team except the oldest member (graduated) was replaced with a younger student. If the average team member age was the same for both years, how many years younger was the new member than the old member who was replaced?

(A) 1

(B) 2

(C) 3

$\boxed{\text{(D)}}$ 4

(E) 5

SOLN

$$\frac{a_1+a_2+a_3+a_4+a_{\text{old}}}{5} = \frac{(a_1+a_2+a_3+a_4+4)+a_{\text{new}}}{5}$$

□

37. A regular decagon, side length 1, is inscribed in a circle. Find the radius of the circle.

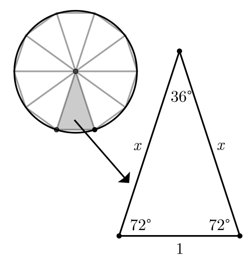
(A) 1

$\boxed{\text{(B)}}$ $\frac{1+\sqrt{5}}{2}$

(C) $\frac{1-\sqrt{5}}{2}$

(D) $\frac{\sqrt{5}}{2}$

(E) $\frac{\sqrt{10}}{2}$



SOLN Divide the decagon into 10 equal isosceles 36°-72°-72° triangles having equal legs length x. From the Laws of Sines and Cosines and the double-angle identity, we get the equations $\frac{x}{2} = \cos 36°$ and $1 = 2x^2 - 2x^2\cos 36°$. Substituting $\frac{x}{2}$ for $\cos 36°$ in the second equation, simplifying, and using synthetic division, we find the solutions $x = 1, \frac{1\pm\sqrt{5}}{2}$. Both 1 and $\frac{1-\sqrt{5}}{2}$ can be discarded as the former would imply $\cos 36° = \frac{1}{2}$ (not true), and the latter is negative. (Inspired by Penrose's dart-kite tessellations; `http://tinyurl.com/kna8sxm`.) □

38. What is the range of possible values for the sum S of angles of a triangle drawn on the surface of a sphere?

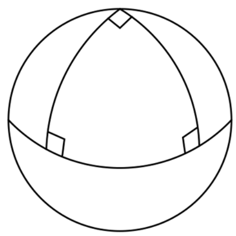
(A) $S = 180°$

(B) $180° < S < 360°$

(C) $180° < S < 720°$

(D) $180° < S < 900°$

(E) $270° < S < 1080°$



SOLN Note that every triangle on a sphere has angle sum greater than 180°. Furthermore, every triangle divides the sphere into two triangular regions that cover the entire sphere. Consider a triangle on a sphere with angles a, b, c summing to something over 180°. The other "outside" triangle has angles $360° - a, 360° - b, 360° - c$, with sum $1080° - (a + b + c) < 900°$.

(From *Heart of Mathematics*, 4th ed., p. 303.) □

39. Adam celebrates only his prime birthdays. Next year there will be a celebration. Adam most recently celebrated his birthdays 5 and 7 years ago. If Adam is currently 78, how many times prior to this year has the preceding information been true?

(A) never

(B) once

(C) twice

(D) thrice

(E) more than three times

SOLN Brute force: list all primes up to 79 and look for the right pattern. Better: recognize that no number ending in an even or 5 is prime. To fit the needed pattern of 3 primes of $x, x + 2, x + 8$, x also cannot end in 3 or 7. So the pattern will always start with a prime that ends in 1 or 9. Can't start with 11, as it would skip 17. So the needed patterns prior to (71, 73, 79) are (29, 31, 37), and (59, 61, 67). □

40. John has a weird way of cutting pie. The first piece is a full half of the pie. The second piece is 1/3 of the remaining half of pie. Then the 3rd piece is 1/4 of the remainder and so on. What fraction of the original pie is the 5th piece?

(A) $\frac{1}{720}$

(B) $\frac{1}{30}$

(C) $\frac{1}{120}$

(D) $\frac{1}{6}$

(E) $\frac{1}{540}$

SOLN The 5th piece is $\frac{1}{6}$ of $\frac{4}{5}$ of $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$ or $\frac{4\cdot3\cdot2}{6\cdot5\cdot4\cdot3\cdot2} = \frac{1}{30}$. □